PERGAMON

International Journal of Solids and Structures 36 (1999) 1925–1947

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

An active control model of laminated piezothermoelastic plate

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Received 15 January 1997; in revised form 18 February 1998

Abstract

After the Hamilton principle for thermo-mechanical–electric coupling problem is derived, the third-order shear deformation theory is extended to encompass piezothermoelastic laminated plates. Based on the velocity feedback control, a model for the active vibration control of laminated plates with piezothermoelastic sensor/actuator is established. An analytical solution is obtained for the case of general forces acting on a simply supported piezothermoelastic laminated plate. Numerical results are presented. The factors that influence the controlled responses of the plate are examined. © 1999 Elsevier Science Ltd. All rights reserved.

Nomenclature

a	length of the plate
b	width of the plate
c_{ij}	transform reduced stiffness constant
D_i	electric displacement in the i axis
E_i	electric field intensity in the i axis
e_{ij}	transformed reduced piezoelectric constant
$f_j^k(z)$	Lagrange interpolation function
H	thickness of laminate
h_1, h_2	thickness of G/E, PVDF lamina
K	total kinetic energy
q_i	heat flow in the i axis
U	total potential energy
u_i	displacement in the i axis
u_0	displacement of a point on the midplane in x axis
v_0	displacement of a point on the midplane in y axis

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PII: S0020–7683(98)00068–7

w_0 displacement of a point on the midplane in z axis
β_i transformed reduced thermal constant
ε_{ij} permitivity matrix
$\hbar$ Helmholtz free energy
θ_0, θ_N temperature increase at the upper and lower surfaces of the plate
ϑ dissipative function
σ_i stress
γ_i strain
η specific entropy
τ_i piezothermoelastic constant
λ_{ij} coefficient of heat conduction
ϕ electric potential
ψ_x, ψ_y rotation of normals to midplane about y and x axes
ω excitation frequency
ω_0 the lowest natural frequency

1. Introduction

Due to the intrinsic thermo-mechanical–electric coupling effects, piezothermoelastic materials have been widely used in engineering structures to detect the responses of the structure by measuring the electric charge, sensing, or to reduce excessive responses by applying additional electric forces or thermal forces, actuating. By integrating the sensing and actuating, it is possible to create so-called intelligent structures and systems that can adapt to or correct for changing operating conditions.

In order to utilize the sensing and actuating properties of piezothermoelastic materials, the interaction between the structure and the smart material must be well understood. One of the earliest and most comprehensive studies concerned with piezoelectric plates is the work of Tiersten (1969), who obtained the dynamic equations and solutions with ignoring thermal effect. Crawley and de Luise (1987), Im and Atluri (1989), and Chandra and Chopra (1993) developed the mechanical models for studying the interaction of piezoelectric patches surface-mounted to beams. Crawley and Lazarus (1991) used surface mounted piezoelectric devices in strain prediction and control of structures. Lee (1990), and Mitchell and Reddy (1995) derived theories for laminated piezoelectric plates using classical plate theory and simple third-order theory (Reddy, 1984), respectively. Wang and Rogers (1991) presented a model based classical plate theory for laminated plates with spatially distributed piezoelectric patches. Chandrashekhara and Agarwal (1993) developed an active vibration control model for laminated plate with piezoelectric layer based on the first-order shear deformation theory. Tzou (1989), and Tzou and Gadre (1989) analyzed thin laminates coupled with shell actuators for distributed vibration control.

It has been recognized that the temperature variation can affect the overall performance of a control system (Tzou and Tseng, 1990). The governing equations of a piezothermoelastic medium were first derived by Mindlin (1961, 1974). Nowacki (1978, 1983) gave general theorems and mathematical models of piezothermoelasticity. Rao and Sunar (1994) developed a finite element formulation of piezothermoelastic media and integrated it with the distributed sensing and control of intelligent structures. Tauchert (1992) applied Nowacki's general theory to a piezothermoelastic laminated plate and obtained static solutions using the classical lamination theory. Tang and Xu (1995) generalized Tauchert's work to dynamic problems of piezothermoelastic laminated plates, the dynamic solutions of the plates were derived for general forced vibrations with simply supported boundary conditions. However, they did not establish the active vibration control model.

Since plates can be considered as fundamental elements of large space structures, video head positioners, microgrippers, etc., the problem of how to detect their motions and control them are two of the questions that naturally arise. In this paper, the Hamilton principle for thermo-mechanical–electric coupling problem is derived first, then the work of Mitchell and Reddy (1995) is extended to encompass piezothermoelastic laminated plates. Based on the velocity feedback control, a model for the active vibration control of laminated plates with piezothermoelastic sensor/actuator is established. An analytical solution is obtained for the case of general forces acting on a simply supported piezothermoelastic laminated plate. Numerical results are presented. The factors that influence the controlled responses of the plate are examined.

2. The Hamilton principle for thermo-mechanical–electric coupling problem

In this paper, the generalized Hamilton principle is applied to derive a set of approximate governing equations for laminated plates with piezothermoelastic laminae. One of the approximations is that the electric field is quasistatic and that the influence of a magnetic field and magnetization is negligible. The other is that deformations are infinitesimal and that electric fields are small. In this section, the Hamilton principle will be derived based on Helmholtz energy.

Let V be the spatial region occupied by the body considered, s be the total surface and $\boldsymbol{n}$ be the outward normal. In a fixed rectangular coordinate system ox_i ($i = 1, 2, 3$), let $\boldsymbol{u}$, ϕ, $\boldsymbol{\sigma}$, $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{q}$ be the displacements, electric potential, stress, electric induction, electric field and heat flow, respectively, and $\theta = T - T_0$ is the temperature rise from the stress-free reference temperature T_0. Let η, Ξ and t be the specific entropy, heat source and time, respectively. Field equation in the domain $V \times [0, \infty)$, the governing equations are:

The strain–displacement and electric field-potential relations:

$$\gamma_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$
$$E_i = -\phi_{,i} \tag{1}$$

The equations of motion and the charge equation of electrostatics

$$\sigma_{ij,j} = \rho \ddot{u}_i$$
$$D_{i,j} = 0 \tag{2}$$

The energy equation

$$q_{i,i} + T_0 \dot{\eta} = \Xi \tag{3}$$

the constitutive equations

$$\sigma_{ij} = c_{ijkl}\gamma_{kl} - e_{sji}E_s - \beta_{ij}\theta$$

$$D_i = \varepsilon_{is}E_s + e_{irs}\gamma_{rs} + \tau_i\theta$$

$$\eta = \beta_{ij}\gamma_{ij} + \tau_i E_i + C\frac{\theta}{T_0} \tag{4}$$

the heat conduction equations

$$q_i = -\lambda_{ij}\theta_{,j} \tag{5}$$

where c_{ijkl}, e_{sji}, ε_{ij}, β_{ij}, τ_i, λ_{ij} and C as the elasticity constants, piezoelectricity, permitivity, stress-temperature coefficients, piezothermoelasticity, coefficients of heat conduction and heat capacity.

The mechanical boundary conditions are take to be

$$u_i = \bar{u}_i$$

$$P_i = \sigma_{ij}n_j = \bar{P}_i \tag{6}$$

where the overbar $^{-}$ indicates the prescribed values on the boundary. The electric boundary conditions are

$$\phi = \bar{\phi}$$

$$\Omega = -n_i D_i = \bar{\Omega}$$

and the thermal boundary conditions are

$$\theta = \bar{\theta}$$

$$Q = q_i n_i = \bar{Q}$$

Initial conditions are

$$u_i(x_1, x_2, x_3, 0) = u_{0i}(x_1, x_2, x_3)$$

$$\dot{u}_i(x_1, x_2, x_3, 0) = \dot{u}_{0i}(x_1, x_2, x_3)$$

$$\theta(x_1, x_2, x_3, 0) = \theta_0(x_1, x_2, x_3)$$

$$\dot{\theta}(x_1, x_2, x_3, 0) = \dot{\theta}_0(x_1, x_2, x_3)$$

$$\phi(x_1, x_2, x_3, 0) = \phi_0(x_1, x_2, x_3) \tag{7}$$

The total kinetic energy is

$$K = \int_V \frac{1}{2}\rho\dot{u}_i\dot{u}_j\,\mathrm{d}V$$

The total potential energy is

$$U = \int_V [\hbar(\gamma_{ij}, \theta, E_i) + \eta\theta]\,\mathrm{d}V - \int_{S_P} \bar{P}_i u_i\,\mathrm{d}S + \int_V (R - h\theta)\,\mathrm{d}V + \int_{S_Q} \bar{Q}\theta\,\mathrm{d}S + \int_{S_\Omega} \bar{\Omega}\phi\,\mathrm{d}S$$

$$\hbar = \frac{1}{2}c_{ijkl}\gamma_{ij}\gamma_{kl} - e_{sji}E_s\gamma_{ij} - \frac{1}{2}\varepsilon_{is}E_sE_i - \beta_{ij}\gamma_{ij}\theta - \tau_i E_i\theta - \frac{C\theta^2}{2T_0}$$

$$R = \tfrac{1}{2}k_{ij}\theta_{,i}\theta_{,j}$$

where $\hbar$ denotes the Helmholtz free energy.

The dissipative function is

$$\vartheta = \int_V T_0\dot{\eta}\theta \,\mathrm{d}V$$

Then, the Hamilton's principle can be obtained as

$$\delta\int_{t_1}^{t_2}(K-U-\vartheta)\,\mathrm{d}t = \int_{t_1}^{t_2}\left\{\int_V\left[(\sigma_{ij,j}-\rho\ddot{u}_i)\delta u_i-\left(\frac{\partial\hbar}{\partial\theta}+\eta\right)\delta\theta-(q_{i,i}+T_0\dot{\eta}-\Xi)\delta\theta+D_{i,i}\delta\phi\right]\mathrm{d}V\right.$$
$$\left.+\int_{S_Q}(Q-\bar{Q})\delta\theta\,\mathrm{d}s-\int_{S_P}(P_i-\bar{P}_i)\delta u_i\,\mathrm{d}S-\int_{S_\Omega}(\Omega-\bar{\Omega})\delta\phi\,\mathrm{d}S\right\}\mathrm{d}t = 0 \quad (8)$$

This principle is analogous to that in Nowacki (1978), which can apply to all types of infinitesimal deformation and small electric field plates. The control equations (1)–(3) and boundary conditions (6) can be derived from the above equation.

3. Basic equations

In this section, we generalize Reddy's work (Mitchell and Reddy, 1995) to piezothermoelastic laminated plates. Consider a hybrid piezothermoelastic laminated plate, which is symmetrically laminated, as shown in Fig. 1. The principle material directions are assumed to coincide with the coordinates of the problem being analyzed. The constitutive relationship in vectorial form for a piezothermoelastic material with orthorhombic mm2 symmetry can be written as follows (see e.g. Mindlin, 1974; Rao and Sunar, 1994)

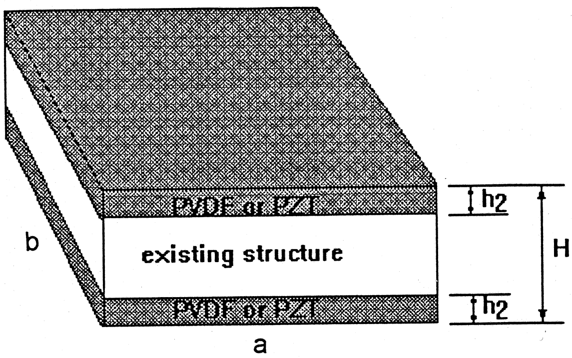

Fig. 1. Laminate configuration.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{22} & c_{23} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{55} & 0 \\ & & & & & c_{66} \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & e_{33} \\ 0 & e_{24} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \phi}{\partial z} \end{Bmatrix} - \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \theta \tag{9}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{Bmatrix} - \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \phi}{\partial z} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \tau_3 \end{Bmatrix} \theta \tag{10}$$

where σ_i, γ_i, D_i, ϕ and θ represent the stress, engineering strain, electric induction, electric potential and temperature rise, respectively. c_{ij} are the transformed reduced stiffness, e_{ij} are transformed piezoelectric constants and β_i are transformed thermal constants that have been adjusted to accommodate the plane stress approximation.

In the third-order shear deformation theory of Reddy (Reddy, 1984; Mitchell and Reddy, 1995), the displacement field is assumed to be of the form

$$\begin{aligned} u_1(x,y,z,t) &= u_0(x,y,t) + g_1(z)\psi_x(x,y,t) - g_2(z)\frac{\partial w_0}{\partial x} \\ u_2(x,y,z,t) &= v_0(x,y,t) + g_1(z)\psi_y(x,y,t) - g_2(z)\frac{\partial w_0}{\partial y} \\ u_3(x,y,z,t) &= w_0(x,y,t) \end{aligned} \tag{11}$$

Here, u_0, v_0, w_0 are the displacement of a point on the midplane of the laminate, and ψ_x, ψ_y denote the rotations of a transverse normal at $z = 0$ about the y and x axes, respectively. The functions $g_1(z)$ and $g_2(z)$ are given as

$$g_1(z) = z - \frac{4z^3}{3H^2}, \quad g_2(z) = \frac{4z^3}{3H^2} \tag{12}$$

where H is the total laminate thickness and z is assumed to be measured from the laminate geometric midplane, and the electric potential function ϕ on a discrete layer is written as follows

$$\phi^k(x,y,z,t) = \sum_{j=1}^{n} f_j^k(z)\varphi_j^k(x,y,t) \tag{13}$$

where $f_j^k(z)$ are taken to be Lagrange interpolation functions. Using these definitions, it is easy to verify that the transverse shear stresses σ_4 and σ_5 are zero at the upper and lower surfaces of the plate. The plate is subject to the linear temperature variation (Tauchert, 1992)

$$\theta(z) = \frac{\theta_0 + \theta_N}{2} + \frac{\theta_N - \theta_0}{H} z \tag{14}$$

where θ_0 and θ_N denote temperature amplitude increases at the upper and lower surfaces of the plate, respectively, which can be induced by infrared radiation.

Proceeding as described above, eqn (8) is expanded, giving

$$\begin{aligned} 0 &= \delta K + \delta U + \delta E - \delta V \\ &= \int_0^{t_0} \int_V \rho(\dot{u}_1 \delta \dot{u}_1 + \dot{u}_2 \delta \dot{u}_2 + \dot{u}_3 \delta \dot{u}_3)\, \mathrm{d}V\, \mathrm{d}t \\ &\quad - \int_0^{t_0} \int_V \{\sigma_1 \delta\gamma_1 + \sigma_2 \delta\gamma_2 + \sigma_3 \delta\gamma_3 + 2\sigma_4 \delta\gamma_4 + 2\sigma_5 \delta\gamma_5 + 2\sigma_6 \delta\gamma_6\}\, \mathrm{d}V\, \mathrm{d}t \\ &\quad - \int_0^{t_0} \int_V (D_1 \delta E_1 + D_2 \delta E_2 + D_3 \delta E_3)\, \mathrm{d}V\, \mathrm{d}t \\ &\quad + \int_0^{t_0} \int_S \{P_1 \delta u_1 + P_2 \delta u_2 + P_3 \delta u_3 + q \delta\phi\}\, \mathrm{d}S\, \mathrm{d}t \end{aligned} \tag{15}$$

The first integral denotes the virtual kinetic energy, the second denotes the virtual work done by internal forces, the third represents the contribution of the electric field and the fourth denotes the virtual potential energy due to applied forces. The equilibrium equations for the laminated piezothermoelectric plate are obtained the same as those of the plates of composite materials (Reddy, 1984; Mitchell and Reddy, 1995)

$$-\left(I_1 \ddot{u}_0 + I_2 \ddot{\psi}_x - I_3 \frac{\partial \ddot{w}_0}{\partial x}\right) + \frac{\partial \bar{N}_1}{\partial x} + \frac{\partial \bar{N}_6}{\partial y} = 0$$

$$-\left(I_1 \ddot{v}_0 + I_2 \ddot{\psi}_y - I_3 \frac{\partial \ddot{w}_0}{\partial y}\right) + \frac{\partial \bar{N}_2}{\partial y} + \frac{\partial \bar{N}_6}{\partial x} = 0$$

$$-\left[I_1 \ddot{w}_0 + \frac{\partial}{\partial x}\left(I_3 \ddot{u}_0 + I_5 \ddot{\psi}_x - I_6 \frac{\partial \ddot{w}_0}{\partial x}\right) + \frac{\partial}{\partial y}\left(I_3 \ddot{v}_0 + I_5 \ddot{\psi}_y - I_6 \frac{\partial \ddot{w}_0}{\partial y}\right)\right]$$

$$+ t_3 + \frac{\partial^2 \bar{P}_1}{\partial x^2} + \frac{\partial^2 \bar{P}_2}{\partial y^2} + 2\frac{\partial^2 \bar{P}_6}{\partial x\, \partial y} + \frac{\partial \bar{Q}_4}{\partial y} + \frac{\partial \bar{Q}_5}{\partial x} = 0$$

$$-\left[\left(I_2 \ddot{u} + I_4 \ddot{\psi}_x - I_5 \frac{\partial \ddot{w}_0}{\partial x}\right)\right] + \frac{\partial \bar{M}_1}{\partial x} + \frac{\partial \bar{M}_6}{\partial y} - \bar{Q}_5 = 0$$

$$-\left[\left(I_2 \ddot{v}_0 + I_4 \ddot{\psi}_y - I_5 \frac{\partial \ddot{w}_0}{\partial y}\right)\right] + \frac{\partial \bar{M}_2}{\partial y} + \frac{\partial \bar{M}_6}{\partial x} - \bar{Q}_4 = 0$$

$$\frac{\partial \bar{P}_\alpha^{jk}}{\partial x_\alpha} - \bar{G}_3^{jk} = 0 \quad (\alpha = 1, 2, j = 1, 2) \tag{16}$$

where $(\bar{N}_i, \bar{M}_i, \bar{P}_i, \bar{Q}_i)$ are the stress resultants

$$\bar{N}_i = \int_{-H/2}^{H/2} \sigma_i \, \mathrm{d}z, \quad \bar{M}_i = \int_{-H/2}^{H/2} \sigma_i g_1(z) \, \mathrm{d}z, \quad \bar{P}_i = \int_{-H/2}^{H/2} \sigma_i g_2(z) \, \mathrm{d}z, \quad i = 1, 2, 6$$

$$\bar{Q}_i = \int_{-H/2}^{H/2} \sigma_i \frac{\mathrm{d}g_1}{\mathrm{d}z} \, \mathrm{d}z, \quad i = 4, 5$$

$(P_\alpha^{jk}, G_3^{jk})$ and $l_1, l_2, \ldots, l_6$ are defined, respectively, as

$$\bar{P}_\alpha^{jk} = \int_{z_{k-1}}^{z_k} D_\alpha f_j^k(z) \, \mathrm{d}z, \quad \bar{G}_3^{jk} = \int_{z_{k-1}}^{z_k} D_3 \frac{\mathrm{d}f_j^k}{\mathrm{d}z} \mathrm{d}z$$

$$I_1 = \int_{-H/2}^{H/2} \rho \, \mathrm{d}z, \quad I_2 = \int_{-H/2}^{H/2} \rho g_1 \, \mathrm{d}z, \quad I_3 = \int_{-H/2}^{H/2} \rho g_2 \, \mathrm{d}z$$

$$I_4 = \int_{-H/2}^{H/2} \rho g_1 g_1 \, \mathrm{d}z, \quad I_5 = \int_{-H/2}^{H/2} \rho g_1 g_2 \, \mathrm{d}z, \quad I_6 = \int_{-H/2}^{H/2} \rho g_2 g_2 \, \mathrm{d}z$$

Moreover, for symmetric laminated, by directly integrating it is readily obtained that $I_2 = I_3 = 0$.

Now, the stress results $(\bar{N}_i, \bar{M}_i, \bar{P}_i, \bar{Q}_i)$ are decomposed into three parts

$$\begin{aligned}
\bar{N}_i &= N_i + N_i^P - N_i^\theta \\
\bar{M}_i &= M_i + M_i^P - M_i^\theta \\
\bar{P}_i &= P_i + P_i^P - P_i^\theta \\
\bar{Q}_i &= Q_i + Q_i^P - Q_i^\theta
\end{aligned} \tag{17}$$

The first part is when the piezoelectric and thermal effects are not present. The second part is due to the piezoelectric effect and the third part is due to the thermal effect. The first part is obtained by integrating eqns (9) and (10) with respect to the thickness coordinate z. The resulting expression for symmetric laminated plate takes the form

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ & A_{22} & 0 \\ & & A_{66} \end{bmatrix} \times \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} \tag{18}$$

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_6 \\ P_1 \\ P_2 \\ P_6 \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & 0 & \bar{B}_{11} & \bar{B}_{12} & 0 \\ & B_{22} & 0 & \bar{B}_{12} & \bar{B}_{22} & 0 \\ & & B_{66} & 0 & 0 & \bar{B}_{66} \\ & & & D_{11} & D_{12} & 0 \\ & & & & D_{22} & 0 \\ & & & & & D_{66} \end{bmatrix} \times \begin{Bmatrix} \dfrac{\partial \psi_x}{\partial x} \\ \dfrac{\partial \psi_y}{\partial y} \\ \dfrac{\partial \psi_x}{\partial y} + \dfrac{\partial \psi_y}{\partial x} \\ -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x\,\partial y} \end{Bmatrix} \tag{19}$$

$$\begin{Bmatrix} Q_4 \\ Q_5 \end{Bmatrix} = \begin{bmatrix} F_{44} & 0 \\ 0 & F_{55} \end{bmatrix} \times \begin{Bmatrix} \psi_y + \dfrac{\partial w_0}{\partial y} \\ \psi_x + \dfrac{\partial w_0}{\partial x} \end{Bmatrix} \tag{20}$$

where the laminate stiffness A_{ij}, B_{ij} and so on are defined as

$$B_{ij} = \int_{-H/2}^{H/2} g_1 g_1 c_{ij}\,\mathrm{d}z, \quad \bar{B}_{ij} = \int_{-H/2}^{H/2} g_1 g_2 c_{ij}\,\mathrm{d}z$$

$$A_{ij} = \int_{-H/2}^{H/2} c_{ij}\,\mathrm{d}z, \quad D_{ij} = \int_{-H/2}^{H/2} g_2 g_2 c_{ij}\,\mathrm{d}z \quad (i,j = 1,2,6)$$

$$F_{44} = \int_{-H/2}^{H/2} c_{44} \left(\frac{\mathrm{d}g_1}{\mathrm{d}z}\right)^2 \mathrm{d}z, \quad F_{22} = \int_{-H/2}^{H/2} c_{55} \left(\frac{\mathrm{d}g_1}{\mathrm{d}z}\right)^2 \mathrm{d}z$$

and the stress resultants due to the piezoelectric effect can be defined as

$$N_1^P = \sum_{k=1}^{m} e_{31}^k \left(\sum_{j=1}^{n} \eta_1^{jk} \varphi_j^k\right), \quad N_2^P = \sum_{k=1}^{m} e_{32}^k \left(\sum_{j=1}^{n} \eta_1^{jk} \varphi_j^k\right), \quad N_6^P = 0$$

$$M_1^P = \sum_{k=1}^{m} e_{31}^k \left(\sum_{j=1}^{n} \eta_2^{jk} \varphi_j^k\right), \quad M_2^P = \sum_{k=1}^{m} e_{32}^k \left(\sum_{j=1}^{n} \eta_2^{jk} \varphi_j^k\right), \quad M_6^P = 0$$

$$P_1^P = \sum_{k=1}^{m} e_{31}^k \left(\sum_{j=1}^{n} \eta_3^{jk} \varphi_j^k\right), \quad P_2^P = \sum_{k=1}^{m} e_{32}^k \left(\sum_{j=1}^{n} \eta_3^{jk} \varphi_j^k\right), \quad P_6^P = 0$$

$$Q_4^P = \sum_{k=1}^{m} e_{24}^k \left(\sum_{j=1}^{n} \Delta_4^{jk} \frac{\partial \varphi_j^k}{\partial y} \right), \quad Q_5^P = \sum_{k=1}^{m} e_{15}^k \left(\sum_{j=1}^{n} \Delta_4^{jk} \frac{\partial \varphi_j^k}{\partial x} \right) \tag{21}$$

where

$$\eta_1^{jk} = \int_{z_{k-1}}^{z_k} \frac{\mathrm{d}f_j^k}{\mathrm{d}z} \mathrm{d}z, \quad \eta_2^{jk} = \int_{z_{k-1}}^{z_k} g_1 \frac{\mathrm{d}f_j^k}{\mathrm{d}z} \mathrm{d}z$$

$$\eta_3^{jk} = \int_{z_{k-1}}^{z_k} g_2 \frac{\mathrm{d}f_j^k}{\mathrm{d}z} \mathrm{d}z, \quad \Delta_4^{jk} = \int_{z_{k-1}}^{z_k} \frac{\mathrm{d}g_1}{\mathrm{d}z} f_j^k \, \mathrm{d}z$$

In addition the stress resultants due to the thermal effects are defined as

$$\begin{Bmatrix} N_1^\theta \\ N_2^\theta \\ N_6^\theta \end{Bmatrix} = \sum_{k=1}^{m} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{bmatrix} \theta \, \mathrm{d}z, \quad \begin{Bmatrix} M_1^\theta \\ M_2^\theta \\ M_6^\theta \end{Bmatrix} = \sum_{k=1}^{m} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{bmatrix} g_1 \theta \, \mathrm{d}z$$

$$\begin{Bmatrix} P_1^\theta \\ P_2^\theta \\ P_6^\theta \end{Bmatrix} = \sum_{k=1}^{m} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{bmatrix} g_2 \theta \, \mathrm{d}z, \quad Q_4^\theta = Q_5^\theta = 0 \tag{22}$$

The boundary conditions are given as

$$\begin{aligned}
u_0 &\quad \text{or } n_x \bar{N}_1 + n_y \bar{N}_6 \\
v_0 &\quad \text{or } n_y \bar{N}_2 + n_x \bar{N}_6 \\
w_0 &\quad \text{or } n_x \left(I_5 \ddot{\psi}_x - I_6 \frac{\partial \ddot{w}_0}{\partial x} \right) + n_y \left(I_5 \ddot{\psi}_y - I_6 \frac{\partial \ddot{w}_0}{\partial y} \right) + Q_n \\
\frac{\partial w_0}{\partial x} &\quad \text{or } n_x \bar{P}_1 + n_y \bar{P}_6 \\
\frac{\partial w_0}{\partial y} &\quad \text{or } n_y \bar{P}_2 + n_x \bar{P}_6 \\
\psi_x &\quad \text{or } -n_x \bar{M}_1 - n_y \bar{M}_6 \\
\psi_y &\quad \text{or } -n_y \bar{M}_2 - n_x \bar{M}_6
\end{aligned} \tag{23}$$

where

$$Q_n = -n_x \frac{\partial \bar{P}_1}{\partial x} - n_y \frac{\partial \bar{P}_2}{\partial y} - n_x \frac{\partial \bar{P}_6}{\partial y} - n_y \frac{\partial \bar{P}_6}{\partial x} - n_y \bar{Q}_4 - n_x \bar{Q}_5$$

In this paper we study simply supported and symmetric cross-ply plates, the equations of motion for the displacement can be expressed as

$$A_{11}\frac{\partial^2 u_0}{\partial x^2}+(A_{12}+A_{66})\frac{\partial^2 v_0}{\partial x\,\partial y}+A_{66}\frac{\partial^2 u_0}{\partial y^2}+\sum_{k=1}^{m}e_{31}^k\left(\sum_{j=1}^{n}\eta_1^{jk}\frac{\partial\varphi_j^k}{\partial x}\right)=I_1\ddot{u}_0+\frac{\partial N_1^\theta}{\partial x}$$

$$A_{22}\frac{\partial^2 v_0}{\partial y^2}+(A_{12}+A_{66})\frac{\partial^2 u_0}{\partial x\,\partial y}+A_{66}\frac{\partial^2 v_0}{\partial x^2}+\sum_{k=1}^{m}e_{32}^k\left(\sum_{j=1}^{n}\eta_1^{jk}\frac{\partial\varphi_j^k}{\partial y}\right)=I_1\ddot{v}_0+\frac{\partial N_2^\theta}{\partial y}$$

$$-D_{11}\frac{\partial^4 w_0}{\partial x^4}-(4D_{66}+2D_{12})\frac{\partial^4 w_0}{\partial x^2\,\partial y^2}-D_{22}\frac{\partial^4 w_0}{\partial y^4}+F_{44}\frac{\partial^2 w_0}{\partial y^2}+F_{55}\frac{\partial^2 w_0}{\partial x^2}+\bar{B}_{11}\frac{\partial^3\psi_x}{\partial x^3}$$

$$+(\bar{B}_{12}+2\bar{B}_{66})\frac{\partial^3\psi_x}{\partial x\,\partial y^2}+F_{55}\frac{\partial\psi_x}{\partial x}+\bar{B}_{22}\frac{\partial^3\psi_y}{\partial y^3}+(\bar{B}_{12}+2\bar{B}_{66})\frac{\partial^3\psi_y}{\partial y\,\partial x^2}+F_{44}\frac{\partial\psi_y}{\partial y}$$

$$+\sum_{k=1}^{m}\left[e_{31}^k\sum_{j=1}^{n}\eta_3^{jk}\frac{\partial^2\varphi_j^k}{\partial x^2}+e_{32}^k\sum_{j=1}^{n}\eta_3^{jk}\frac{\partial^2\varphi_j^k}{\partial y^2}\right]+\sum_{k=1}^{m}\left[e_{24}^k\sum_{j=1}^{n}\Delta_4^{jk}\frac{\partial^2\varphi_j^k}{\partial y^2}+e_{15}^k\sum_{j=1}^{n}\Delta_4^{jk}\frac{\partial^2\varphi_j^k}{\partial x^2}\right]$$

$$=I_1\ddot{w}_0+\frac{\partial}{\partial x}\left[I_5\ddot{\psi}_x-I_6\frac{\partial w_0}{\partial x}\right]+\frac{\partial}{\partial y}\left[I_5\ddot{\psi}_y-I_6\frac{\partial w_0}{\partial y}\right]+\frac{\partial^2 P_1^\theta}{\partial x^2}+\frac{\partial^2 P_2^\theta}{\partial y^2}-t_3 \tag{24}$$

which corresponds to the stress resultants equilibrium equations, and

$$-\bar{B}_{11}\frac{\partial^3 w_0}{\partial x^3}-(\bar{B}_{12}+2\bar{B}_{66})\frac{\partial^3 w_0}{\partial x\,\partial y^2}-F_{55}\frac{\partial w_0}{\partial x}+B_{11}\frac{\partial^2\psi_x}{\partial x^2}+B_{66}\frac{\partial^2\psi_x}{\partial y^2}-F_{55}\psi_x$$

$$+(B_{12}+B_{66})\frac{\partial^2\psi_y}{\partial x\,\partial y}+\sum_{k=1}^{m}\left[e_{31}^k\sum_{j=1}^{n}\eta_2^{jk}\frac{\partial\varphi_j^k}{\partial x}-e_{15}^k\sum_{j=1}^{n}\Delta_4^{jk}\frac{\partial\varphi_j^k}{\partial x}\right]=I_4\ddot{\psi}_x-I_5\frac{\partial\ddot{w}_0}{\partial x}+\frac{\partial M_1^\theta}{\partial x}$$

$$-\bar{B}_{22}\frac{\partial^3 w_0}{\partial y^3}-(\bar{B}_{12}+2\bar{B}_{66})\frac{\partial^3 w_0}{\partial x^2\,\partial y}-F_{44}\frac{\partial w_0}{\partial y}+B_{66}\frac{\partial^2\psi_y}{\partial x^2}+B_{22}\frac{\partial^2\psi_y}{\partial y^2}-F_{44}\psi_y$$

$$+(B_{12}+B_{66})\frac{\partial^2\psi_x}{\partial x\,\partial y}+\sum_{k=1}^{m}\left[e_{32}^k\sum_{j=1}^{n}\eta_2^{jk}\frac{\partial\varphi_j^k}{\partial y}-e_{24}^k\sum_{j=1}^{n}\Delta_4^{jk}\frac{\partial\varphi_j^k}{\partial y}\right]=I_4\ddot{\psi}_y-I_5\frac{\partial\ddot{w}_0}{\partial y}+\frac{\partial M_2^\theta}{\partial y} \tag{25}$$

which corresponds to the moment resultants equilibrium conditions. It can be seen that the thermal effect and electric effect are explicitly included in eqns (24) and (25).

For sensor purposes, there exist two conditions: one is the close circuit condition (Lee, 1990) and the other is the open-circuit (Tzou et al., 1993). In this paper, the piezothermoelastic lamine is in the open-circuit condition, the following equation is adopted (Mitchell and Reddy, 1995),

$$G_3^{jk}=\int_{z_{k-1}}^{z_k}\left\{e_{31}\gamma_1+e_{32}\gamma_2-\varepsilon_{33}\left(\sum_{i=1}^{n}\frac{\mathrm{d}f_i^k}{\mathrm{d}z}\varphi_i^k\right)+\tau_3\theta\right\}\frac{\mathrm{d}f_j^k}{\mathrm{d}z}\,\mathrm{d}z=0 \tag{26}$$

Based on the thin lamina assumption, one linear interpolation function is used for the piezothermoelastic lamina, substituting it into the aforesaid equation, and integrating through the thickness of the kth lamina gives

$$V^k = \frac{1}{\varepsilon_{33}} \int_{z_{k-1}}^{z_k} \{e_{31}\gamma_1 + e_{32}\gamma_2 + \tau_3\theta\} \, \mathrm{d}z \tag{27}$$

where $V^k = \varphi_2^k - \varphi_1^k$. The left side of eqn (27) is the output voltage of the sensor, caused by the strain and therm in the right side.

4. Non-damping vibration of a simply supported plate

To obtain meaningful solutions, in this section a laminated plate with simply supported boundary conditions is studied. The hybrid laminate is a six-layer hybrid laminate by adding two piezothermoelastic layers symmetrically to a four-layer graphite/epoxy substrate, which is a symmetric cross-ply panel having ply angles $[0^\circ/90^\circ]_s$ and ply thickness h_1 (Fig. 1). Let H and h_2 denote the total plate thickness and the piezothermoelastic layer thickness, respectively. The piezothermoelastic layers are subjected to an applied electric potential ϕ_0. Material properties of the laminae are given in the Appendix.

Employing the method analogous to (Mitchell and Reddy, 1995) each piezothermoelastic layer can be regarded as two mathematical layers with equal thickness, whose electric potential can be modeled using linear Lagrange elements.

Then, the electric potentials can be written as

$$\varphi_1^{L_1} = \phi_0, \quad \varphi_2^{L_1} = \varphi_1^{L_2} = \phi_L, \quad \varphi_2^{L_2} = 0$$

$$\varphi_1^{U_1} = 0, \quad \varphi_2^{U_1} = \varphi_1^{U_2} = \phi_U, \quad \varphi_2^{U_2} = \phi_0$$

where the superscript 'L_1' means the lower half of the lower piezothermoelastic layers, 'L_2' means the upper half of the lower piezothermoelastic layer, 'U_1' means the lower half of the upper piezothermoelastic layer and 'U_2' means the upper half of the upper piezothermoelastic layer. The subscript '1' represents the electric potential at the lower surface of the corresponding layer and '2' the electric potential at the upper surface.

The solution for simply supported laminates has the following form:

$$u_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn} \cos \alpha_m x \sin \beta_n y \, \mathrm{e}^{i\omega t}$$

$$v_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{mn} \sin \alpha_m x \cos \beta_n y \, \mathrm{e}^{i\omega t}$$

$$w_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \alpha_m x \sin \beta_n y \, \mathrm{e}^{i\omega t}$$

$$\psi_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn}^x \cos \alpha_m x \sin \beta_n y \, \mathrm{e}^{i\omega t}$$

$$\psi_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn}^y \sin \alpha_m x \cos \beta_n y \, \mathrm{e}^{i\omega t}$$

$$\phi_L = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \phi_{mn}^L \sin\alpha_m x \sin\beta_n y\, \mathrm{e}^{i\omega t}$$

$$\phi_U = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \phi_{mn}^U \sin\alpha_m x \sin\beta_n y\, \mathrm{e}^{i\omega t} \tag{28}$$

where $\alpha_m = (m\pi/a)$, $\beta_n = (n\pi/b)$ and ω is the excitation frequency.

The plate is subjected to the harmonic mechanical force

$$t_3(x,y,z,t) = t_3(x,y)\,\mathrm{e}^{i\omega t} \tag{29}$$

The temperature variation is

$$\theta(x,y,z,t) = \left(\frac{\theta_0+\theta_N}{2} - \frac{\theta_0-\theta_N}{H}z\right)\mathrm{e}^{i\omega t} \tag{30}$$

The electric potential is

$$\phi_0(x,y,z,t) = \phi_0(x,y)\,\mathrm{e}^{i\omega t} \tag{31}$$

These tractions can be expanded in the double-Fourier series as

$$\theta = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{mn} \sin\alpha_m x \sin\beta_n y\, \mathrm{e}^{i\omega t}$$

$$t_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} t_{mn} \sin\alpha_m x \sin\beta_n y\, \mathrm{e}^{i\omega t}$$

$$\phi_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \phi_{0mn} \sin\alpha_m x \sin\beta_n y\, \mathrm{e}^{i\omega t}$$

$$(N_1^\theta, N_2^\theta, M_1^\theta, M_2^\theta, P_1^\theta, P_2^\theta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (N_{1mn}^\theta, N_{2mn}^\theta, M_{1mn}^\theta, M_{2mn}^\theta, P_{1mn}^\theta, P_{2mn}^\theta) \sin\alpha_m x \sin\beta_n y\, \mathrm{e}^{i\omega t} \tag{32}$$

where

$$(\theta_{mn}, t_{mn}, \phi_{0mn}) = \frac{4}{ab}\int_0^b\int_0^a (\theta, t_3, \phi_0) \sin\alpha_m x \sin\beta_n y\, \mathrm{d}x\, \mathrm{d}y$$

and $(N_{1mn}^\theta, N_{2mn}^\theta, M_{1mn}^\theta, M_{2mn}^\theta, P_{1mn}^\theta, P_{2mn}^\theta)$ can be made out by the same way.

Substituting (28)–(32) into eqns (21), (22), (24) and (25), collecting the coefficients, one obtains

$$\begin{bmatrix} -A_{11}\alpha_m^2 - A_{66}\beta_n^2 + I_1\omega^2 & -(A_{12}+A_{66})\alpha_m\beta_n \\ -(A_{12}+A_{66})\alpha_m\beta_n & -A_{66}\alpha_m^2 - A_{22}\beta_n^2 + I_1\omega^2 \end{bmatrix} \begin{Bmatrix} u_{mn} \\ v_{mn} \end{Bmatrix} = \begin{Bmatrix} \alpha_m N_{1mn}^\theta \\ \beta_n N_{2mn}^\theta \end{Bmatrix} \tag{33}$$

and

$$\mathbf{EU} = \mathbf{T} \tag{34}$$

for any fixed value of m and n, in which

$$\mathbf{U} = [w_{mn} \quad \psi^x_{mn} \quad \psi^y_{mn} \quad \phi^L_{mn} \quad \phi^U_{mn}]^T \tag{35}$$

$$\mathbf{T} = \begin{Bmatrix} [(\Delta_4^{11}+\Delta_4^{28})(e_{24}\beta_n^2+e_{15}\alpha_m^2)+(\eta_3^{11}+\eta_3^{28})(e_{32}\beta_n^2+e_{31}\alpha_m^2)]\phi_{0mn}-P^\theta_{1mn}\alpha_m^2 \\ -P^\theta_{2mn}\beta_m^2-t_{mn} \\ M^\theta_{1mn}\alpha_m-[e_{31}(\eta_2^{11}+\eta_2^{28})-e_{15}(\Delta_4^{11}+\Delta_4^{28})]\alpha_m\phi_{0mn} \\ M^\theta_{2mn}\beta_n-[e_{32}(\eta_2^{11}+\eta_2^{28})-e_{24}(\Delta_4^{11}+\Delta_4^{28})]\beta_n\phi_{0mn} \\ X^{21}_{mn}+X^{12}_{mn}-(\varepsilon_{33}R_1^{21}+\varepsilon_{11}T_1^{21}\alpha_m^2+\varepsilon_{22}T_1^{21}\beta_n^2)\phi_{0mn} \\ X^{27}_{mn}+X^{18}_{mn}-(\varepsilon_{33}R_2^{18}+\varepsilon_{11}T_2^{18}\alpha_m^2+\varepsilon_{22}T_2^{18}\beta_n^2)\phi_{0mn} \end{Bmatrix} \tag{36}$$

where X^{jk}_{mn} are determined by

$$\int_{z_{k-1}}^{z_k} \tau_3\theta\frac{\mathrm{d}f_j^k}{\mathrm{d}z}\mathrm{d}z = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X^{jk}_{mn}\sin\alpha_m x\sin\beta_n y$$

The elements of the coefficient matrix $\boldsymbol{E}$ that is symmetric are given by

$$\begin{aligned}
E_{11} &= -D_{11}\alpha_m^4-(4D_{66}+2D_{12})\alpha_m^2\beta_n^2-D_{22}\beta_n^4+F_{44}\beta_n^2-F_{55}\alpha_m^2+I_1\omega^2+I_6\omega^2(\alpha_m^2+\beta_n^2)\\
E_{12} &= \bar{B}_{11}\alpha_m^3+(\bar{B}_{12}+2\bar{B}_{66})\alpha_m\beta_n^2-F_{55}\alpha_m-I_5\omega^2\alpha_m\\
E_{13} &= \bar{B}_{22}\alpha_m^3+(\bar{B}_{12}+2\bar{B}_{66})\beta_n\alpha_m^2-F_{44}\beta_n-I_5\omega^2\beta_n\\
E_{14} &= -(\Delta_4^{21}+\Delta_4^{12})(e_{24}\beta_n^2+e_{15}\alpha_m^2)-(\eta_3^{21}+\eta_3^{12})(e_{32}\beta_n^2+e_{31}\alpha_m^2)\\
E_{15} &= -(\Delta_4^{27}+\Delta_4^{18})(e_{24}\beta_n^2+e_{15}\alpha_m^2)-(\eta_3^{27}+\eta_3^{18})(e_{32}\beta_n^2+e_{31}\alpha_m^2)\\
E_{22} &= -B_{11}\alpha_m^2-B_{66}\beta_n^2-F_{55}+I_4\omega^2\\
E_{23} &= -(B_{12}+B_{66})\alpha_m\beta_n\\
E_{24} &= [e_{31}(\eta_2^{21}+\eta_2^{12})-e_{15}(\Delta_4^{21}+\Delta_4^{12})]\alpha_m\\
E_{25} &= [e_{31}(\eta_2^{27}+\eta_2^{18})-e_{15}(\Delta_4^{27}+\Delta_4^{18})]\alpha_m\\
E_{33} &= -B_{66}\alpha_m^2-B_{22}\beta_n^2-F_{44}+I_4\omega^2\\
E_{34} &= [e_{32}(\eta_2^{21}+\eta_2^{12})-e_{24}(\Delta_4^{21}+\Delta_4^{12})]\beta_n\\
E_{35} &= [e_{32}(\eta_2^{27}+\eta_2^{18})-e_{24}(\Delta_4^{27}+\Delta_4^{18})]\beta_n\\
E_{44} &= (T_2^{21}+T_1^{12})\varepsilon_{11}\alpha_m^2+(R_2^{21}+R_1^{12})\varepsilon_{33}+(T_2^{21}+T_1^{12})\varepsilon_{22}\beta_n^2\\
E_{45} &= 0\\
E_{55} &= (T_2^{27}+T_1^{18})\varepsilon_{11}\alpha_m^2+(R_2^{27}+R_1^{18})\varepsilon_{33}+(T_2^{27}+T_1^{18})\varepsilon_{22}\beta_n^2
\end{aligned} \tag{37}$$

where

$$T_i^{jk} = \int_{z_{k-1}}^{z_k} f_i^k f_j^k \, \mathrm{d}z, \quad R_i^{jk} = \int_{z_{k-1}}^{z_k} \frac{\mathrm{d}f_i^k}{\mathrm{d}z} \frac{\mathrm{d}f_j^k}{\mathrm{d}z} \, \mathrm{d}z$$

The theory developed herein can be reduced to the classical plate theory when one lets $g_2(z) = 0$.

This theory can also be applied to a laminate plate with spatially distribute piezothermoelastic patches. In this case, the thickness and size of piezothermoelastic patches are assumed to be relatively smaller than those of each lamina, so piezothermoelastic patches can be neglected for calculating the global properties of the laminate.

The electric potential function for distributed piezothermoelastic patches is given as

$$\phi^k(x, y, z, t) = \sum_{j=1}^{n} f_j^k(z) \varphi_j^k(x, y, t) R_k(x, y) \tag{38}$$

where the generalized location function is defined as

$$R_k(x, y) = [H(x - x_1^k) - H(x - x_2^k)] \cdot [H(y - y_1^k) - H(y - y_2^k)]$$

and the Heaviside function, $H(x - x_0)$ is defined as follows

$$\begin{aligned} H(x - x_0) &= 1, \quad z \geqslant z_0 \\ &= 0, \quad z < z_0 \end{aligned} \tag{39}$$

Replacing φ_j^k by $\varphi_j^k R_k$ in eqns (24) and (25), one can obtain the control equations for distributed piezothermoelastic patches.

To investigate the effect of the position of the PVDF layer on the deflection of the laminate, two cases, which are shown in Fig. 2(b and c) are computed in this section. Figure 3 contains the static deflection at the center of the plate for these two cases. When the laminate is only subjected to the thermal field, it is noted that the deflection of case 1 is greater than that of case 2 (as shown in Fig. 3(a)). Similarly, when the laminate is subjected to the electric field alone, Fig. 3(b) shows that the deflection of case 1 is greater than that of case 2. These figures imply that the position of the PVDF layer has an important effect on the performance of the smart structure, which is rational due to the fact that the moment resultants vary with the position of the PVDF layer.

5. Active vibration control of simply supported plates

As shown above, the actuator and sensor equations have been developed separately. For an improved system performance, they have to be used in conjunction to form a closed-loop control system. Based on the velocity feedback control, the model for active vibration control of simply supported laminated plates with piezothermoelastic sensor/actuator is developed.

Consider a hybrid piezothermoelastic laminated plate as shown in Fig. 1. In this section, the sensor eqn (27) is employed. Thus, the output voltage of the sensor can be expressed as

$$V = \frac{1}{\varepsilon_{33}} \left\{ e_{31} \left[\frac{\partial u_0}{\partial x}(z_1 - z_0) + \frac{\partial \psi_x}{\partial x} \int_{z_0}^{z_1} g_1 \, \mathrm{d}z - \frac{\partial^2 w_0}{\partial x^2} \int_{z_0}^{z_1} g_2 \, \mathrm{d}z \right] + e_{32} \left[\frac{\partial v_0}{\partial y}(z_1 - z_0) \right. \right.$$

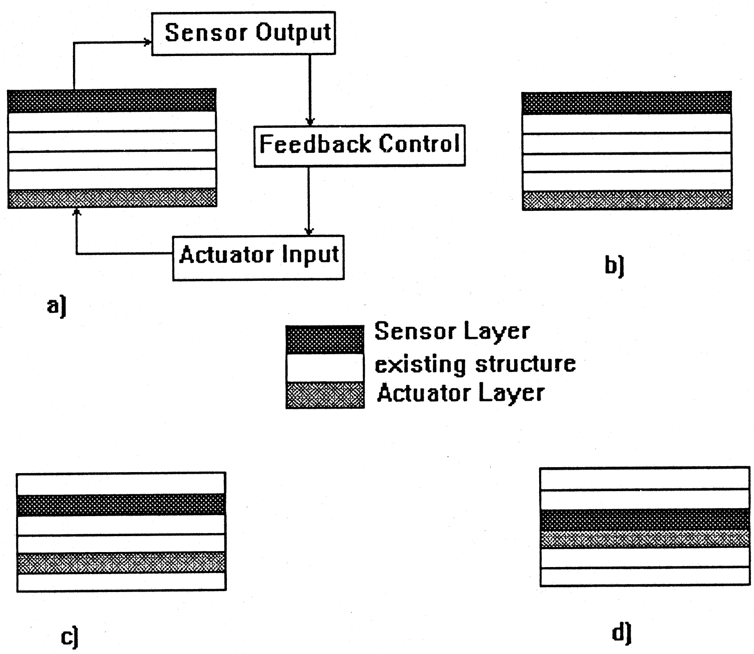

Fig. 2. The control model and three configurations.

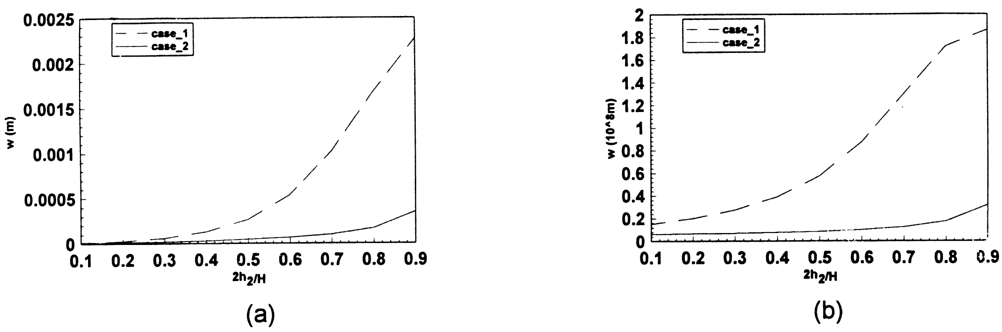

Fig. 3. The effect of position of PVDF on deflection ($a/H = 50$). (a) $\theta_0 = 1°C$, $\theta_N = 0°C$, $\phi = 0$ V. (b) $\theta_0 = \theta_N = 0°C$, $\phi = 1$ V.

$$+\frac{\partial\psi_y}{\partial y}\int_{z_0}^{z_1} g_1\,\mathrm{d}z - \frac{\partial^2 w_0}{\partial y^2}\int_{z_0}^{z_1} g_2\,\mathrm{d}z\Bigg] + \tau_3\int_{z_0}^{z_1}\theta\,\mathrm{d}z\Bigg\} \quad (40)$$

This equation states that the output voltage of the sensor relates the displacement and reflects the vibration of the laminate. To dampen the response of the system, one may use the electric potential to the actuator as the control variable. Since the output voltage of the sensor is accessible, the electric potential applied to the actuator can be expressed as

$$\phi_0 = M\dot{V} = \lambda_1\left[\Omega_1\frac{\partial \dot{u}_0}{\partial x}+\Omega_2\frac{\partial\dot{\psi}_x}{\partial x}-\Omega_3\frac{\partial^2\dot{w}_0}{\partial x}\right]+\lambda_2\left[\Omega_1\frac{\partial \dot{v}_0}{\partial y}+\Omega_2\frac{\partial\dot{\psi}_y}{\partial y}-\Omega_3\frac{\partial^2\dot{w}_0}{\partial y}\right]+\lambda_3\int_{z_0}^{z_1}\dot{\theta}\,\mathrm{d}z \tag{41}$$

where M is the gain to provide feedback control and

$$\lambda_1 = \frac{e_{3i}}{\varepsilon_{33}}M,\quad (i=1,2),\quad \lambda_3 = \frac{\tau_3 M}{\varepsilon_{33}},\quad \Omega_1 = z_1 - z_0,\quad \Omega_2 = \int_{z_0}^{z_1} g_1\,\mathrm{d}z,\quad \Omega_3 = \int_{z_0}^{z_1} g_2\,\mathrm{d}z$$

As the gain is constant with respect to time, this controller is known as a constant gain feedback controller.

For simply supported plate, the responses of the laminate and the excitation are expressed in double Fourier series just like that in Section 4, but replacing $u_{mn}\mathrm{e}^{i\omega t}$ by the general form $u_{mn}(t)$. Thus substituting eqn (41) into eqns (24) and (25), one can obtain the control equation as

$$\mathbf{T}_{mn}\ddot{\mathbf{x}}_{mn}+\mathbf{Q}_{mn}\dot{\mathbf{x}}_{mn}+\mathbf{K}_{mn}\mathbf{x}_{mn} = \mathbf{Z}_{mn} \tag{42}$$

where

$$\mathbf{T}_{mn} = \begin{bmatrix} -I_1-I_6(\alpha_m^2+\beta_n^2) & I_5\alpha_m & I_5\beta_n & 0 & 0 \\ & -I_4 & 0 & 0 & 0 \\ & & -I_4 & 0 & 0 \\ & & & -I_1 & 0 \\ & & & & -I_1 \end{bmatrix}$$

$$\mathbf{x}_{mn} = [w_{mn}(t)\quad \psi^x_{mn}(t)\quad \psi^y_{mn}(t)\quad u_{mn}(t)\quad v_{mn}(t)]^T$$

$$\mathbf{K}^{(ij)}_{mn} = E_{ij}|_{\omega=0}\quad (i,j\leqslant 3),\quad \mathbf{K}^{(54)}_{mn} = \mathbf{K}^{(45)}_{mn} = -(A_{12}+A_{66})\alpha_m\beta_n$$

$$\mathbf{K}^{(44)}_{mn} = -(A_{11}\alpha_m^2+A_{66}\beta_n^2),\quad \mathbf{K}^{(55)}_{mn} = -(A_{66}\alpha_m^2+A_{22}\beta_n^2),\quad \mathbf{K}^{(ij)}_{mn} = 0\quad \text{(for remaining } i,j\text{)}$$

$$\mathbf{Q}_{mn} = \mathbf{F}\cdot\begin{Bmatrix} \Omega_3(\lambda_1\alpha_m^2+\lambda_2\beta_n^2) \\ -\Omega_2\lambda_1\alpha_m \\ -\Omega_2\lambda_2\beta_n \\ -\Omega_1\lambda_1\alpha_m \\ -\Omega_1\lambda_2\beta_n \end{Bmatrix}^T,\quad \mathbf{F} = \begin{Bmatrix} -\eta_3(e_{31}\alpha_m^2+e_{32}\beta_n^2)-\Delta_4(e_{15}\alpha_m^2+e_{24}\beta_n^2) \\ (e_{31}\eta_2-e_{15}\Delta_4)\alpha_m \\ (e_{32}\eta_2-e_{24}\Delta_4)\beta_\alpha \\ e_{31}\eta_1\alpha_m \\ e_{32}\eta_1\beta_n \end{Bmatrix}$$

$$\mathbf{Z}_{mn} = [-P^\theta_{1mn}\alpha_m^2-P^\theta_{2mn}\beta_n^2-t_{mn}\quad M^\theta_{1mn}\alpha_m\quad M^\theta_{2mn}\beta_n\quad N^\theta_{1mn}\alpha_m\quad N^\theta_{2mn}\beta_n]^T-i\omega\lambda_3 G_{mn}\mathbf{F}$$

$$G_{mn} = \int_{z_0}^{z_1}\theta_{mn}\,\mathrm{d}z$$

Here, $\mathbf{Q}_{mn}$ is the feedback gain matrix, i.e. the equivalent damping matrix. The Newmark direct integration scheme (Hughes et al., 1978) can be used to approximate the time derivative in eqn (42).

In this section, these excitations are assumed to be harmonic, they can be expressed as $\mathbf{Z}_{mn}(t) = \mathbf{Z}_{mn}\,e^{i\omega t}$, the steady response of the system can also be expressed $\mathbf{X}_{mn}(t) = \mathbf{X}_{mn}\,e^{i\omega t}$. Substituting this form into eqn (42) and canceling the factor $e^{i\omega t}$ yields

$$[\mathbf{K}_{mn} - \omega^2\mathbf{T}_{mn} + i\omega\mathbf{Q}_{mn}]\mathbf{X}_{mn} = \mathbf{Z}_{mn} \tag{43}$$

Solving this equation can obtain the amplitude $\mathbf{X}_{mn}$.

A numerical example is calculated to show the effect of the property, relative thickness and the position of the piezothermoelastic layer on the performance of the laminate. Attention is again given to the configuration and geometry of the plate shown in Fig. 1. The plate lay-up and material properties are also described in Section 4. In this procedure, three cases are involved which are shown in Fig. 2 (b, c, and d), representing three different positions of the piezothermoelastic sensor/actuator pairs with thickness h_2. The control model is shown in Fig. 2(a). The observation point is at $x = 0.5a$ and $y = 0.5b$ in the plate. The thermal force is considered, and the temperatures on the lower and upper surface are $\theta_0 = 80°\mathrm{C}$ and $\theta_N = 0°\mathrm{C}$, respectively.

In Table 1, we give the lowest natural frequencies ω_0 under different h_2/h_1 for these three cases where the sensor/actuator pairs are PVDF.

Figures 4–6 contain the amplitude of the response w vs the excitation frequency under different gains for all three of the previously mentioned cases and for three different ratios h_2/h_1. The sensor/actuator pairs are PVDF. From these figures, one can find: in case 1, the response is reduced by increasing feedback gains M in the vicinity of ω_0, while ω is far from ω_0, the change of the response by increasing M is very small; when M reaches a certain value, the response becomes a gentle curve whose climax is at $\omega = 0$, moreover, the climax decreases as the thickness of PVDF increases.

In case 2, the result is complicated. Increasing M, the position of the climax moves forward and the climax first decreases, then increases. When ω is far away from the position of the climax, the response decreases as M increases. This means that there exists a certain M that can reduce the response obviously.

In case 3, the response can hardly be suppressed for small piezothermoelastic thickness ratios h_2/h_1. While h_2/h_1 is greater, the effect is similar to that of case 2.

These figures state that the control effect of case 1 is better than that of the others. This provides a rationale to surface mounted piezothermoelastic devices in structural vibration damping.

For PZT sensor/actuator pairs the results are given in Fig. 7 where the piezothermoelastic thickness ratio $h_2/h_1 = 0.02$. The trend resembles that for PVDF. However, due to the fact that the density of PZT is greater than that of graphite/epoxy, the PZT pairs have an obvious effect on the

Table 1
The lowest natural frequencies

h_2/h_1	Case 1	Case 2	Case 3
0.02	167.630	169.447	170.057
1.0	140.111	213.939	242.559
2.0	128.880	254.038	305.370

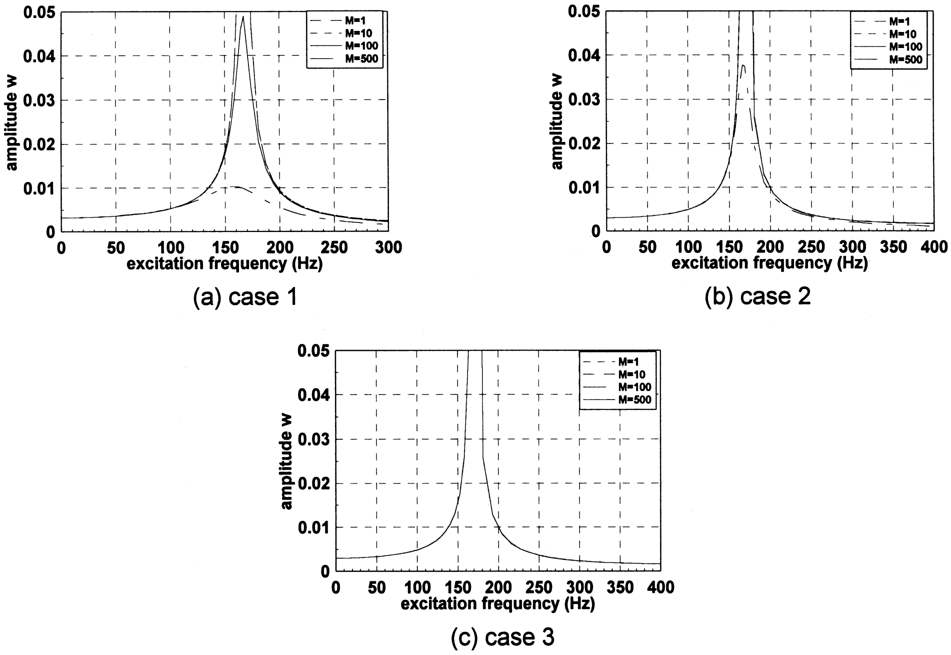

Fig. 4. Amplitude of the response w vs excitation frequency under different feedback gain ($h_2/h_1 = 0.02$, PVDF). (a) Case 1. (b) Case 2. (c) Case 3.

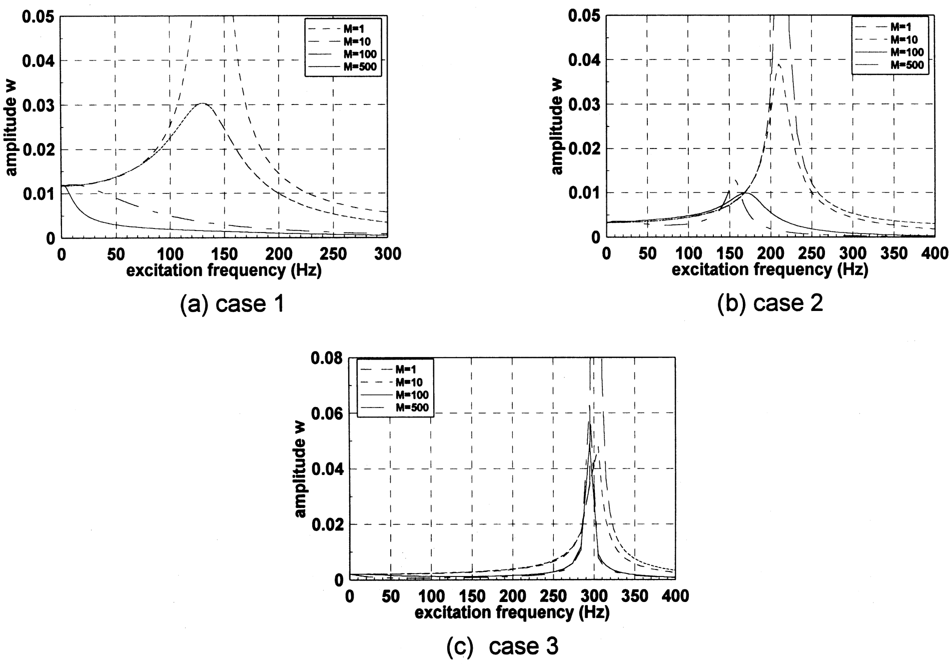

Fig. 5. Amplitude of the response w vs excitation frequency under different feedback gain ($h_2/h_1 = 1$, PVDF). (a) Case 1. (b) Case 2. (c) Case 3.

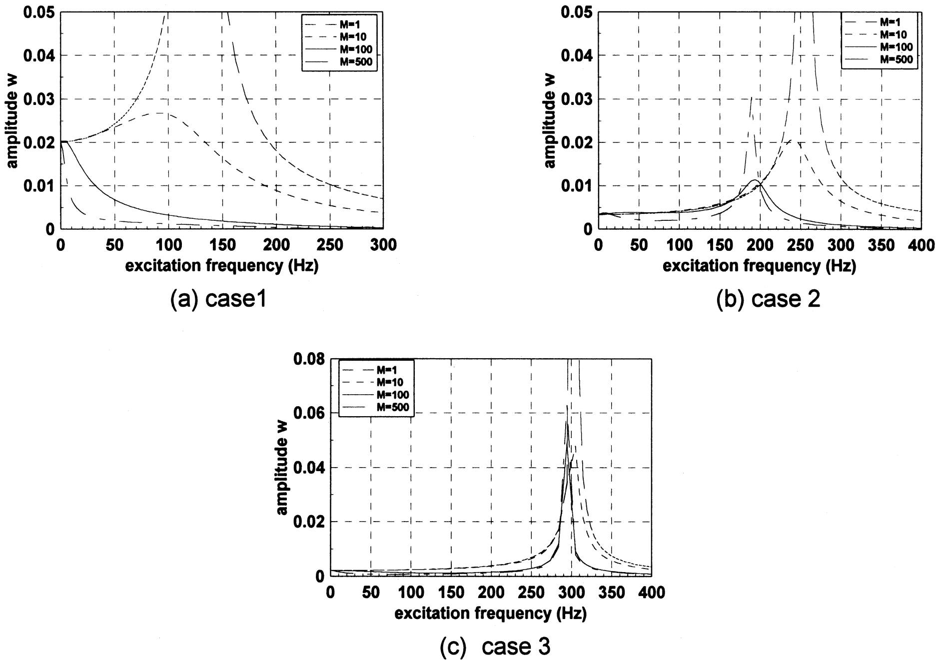

Fig. 6. Amplitude of the response w vs excitation frequency under different feedback gain ($h_2/h_1 = 2$, PVDF). (a) Case 1. (b) Case 2. (c) Case 3.

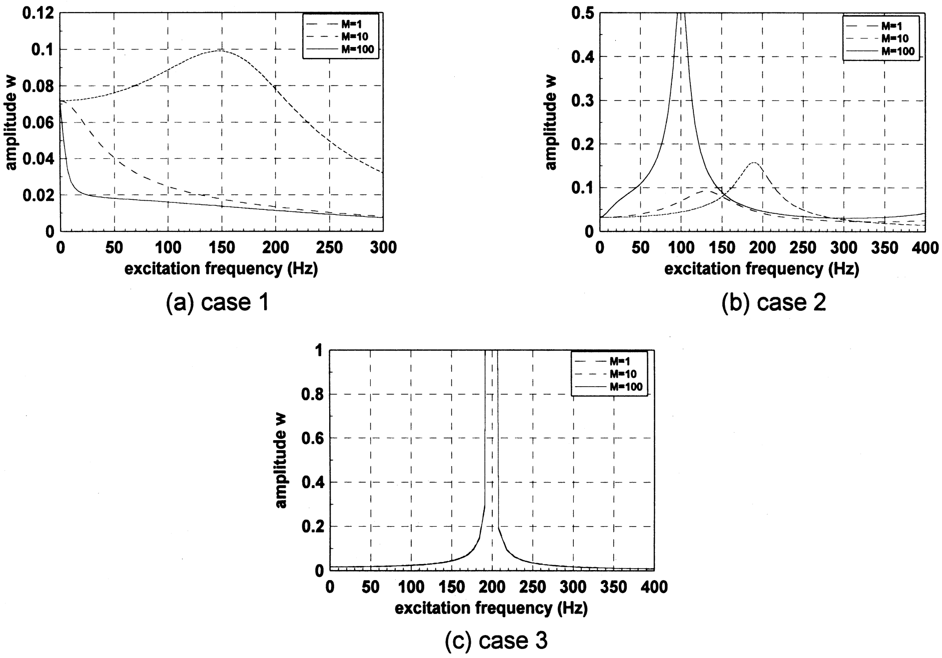

Fig. 7. Amplitude of the response w vs excitation frequency under different feedback gain ($h_2/h_1 = 0.02$, PZT). (a) Case 1. (b) Case 2. (c) Case 3.

global properties of the laminate. Comparing Fig. 7(a) and Fig. 4(a), the reductions by PZT is more obvious than that by PVDF, the climax is suppressed at $M = 10$, but, in factor the response by PZT is greater than that by PVDF at the same M. This implies that the control effect of PZT is no better than that of PVDF.

6. Conclusion

In this paper, the third-order shear deformation theory of Reddy is extended to encompass piezothermoelastic laminated plates. Based on the voltage velocity feedback control the model for the active vibration control of simple supported laminated plates with piezothermoelastic sensor/actuator pairs is established. The results allow us to select the best piezothermoelastic material, position and thickness ratio of the sensor/actuator pairs to control the response of the structure. The harmonic response results show a potential application to actively reduce the harmful effect from thermal forces for a certain control aim.

Appendix

Material properties

Graphite/epoxy
$E_1 = 181$ GPa, $E_2 = E_3 = 10.3$ GPa
$G_{12} = G_{13} = 7.17$ GPa, $G_{23} = 2.87$ GPa
$\mu_{12} = \mu_{13} = 0.28$, $\mu_{23} = 0.33$
$\alpha_1 = 0.02 \times 10^{-6}\ \mathrm{K}^{-1}$, $\alpha_2 = \alpha_3 = 22.5 \times 10^{-6}\ \mathrm{K}^{-1}$
$\rho = 1580\ \mathrm{kg/m}^3$

PZT (Dunn and Taya, 1993):

$$\mathbf{c}^E = \begin{bmatrix} 148 & 76.2 & 74.2 & 0 & 0 & 0 \\ & 148 & 74.2 & 0 & 0 & 0 \\ & & 131 & 0 & 0 & 0 \\ & & & 25.4 & 0 & 0 \\ & & & & 25.4 & 0 \\ & & & & & 35.9 \end{bmatrix} \mathrm{GPa},$$

$$\mathbf{e}^T = \begin{bmatrix} 0 & 0 & -2.1 \\ 0 & 0 & -2.1 \\ 0 & 0 & 9.5 \\ 0 & 9.2 & 0 \\ 9.2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{c/m}^2, \quad \frac{\varepsilon}{\varepsilon_0} = \begin{bmatrix} 460 & 0 & 0 \\ 0 & 460 & 0 \\ 0 & 0 & 235 \end{bmatrix}$$

$$\alpha = 0.9\times10^{-3}\ \mathrm{K}^{-1},\quad \rho = 7600\ \mathrm{kg/m^3}\quad \varepsilon_0 = 8.85\times10^{-12}\ \mathrm{c^2/N\,m^2}$$

PVDF (Varadan et al., 1989)

$$\mathbf{c}^E = \begin{bmatrix} 3.61 & 1.61 & 1.42 & 0 & 0 & 0 \\ & 3.13 & 1.31 & 0 & 0 & 0 \\ & & 1.63 & 0 & 0 & 0 \\ & & & 0.55 & 0 & 0 \\ & & & & 0.59 & 0 \\ & & & & & 0.69 \end{bmatrix} \mathrm{GPa},$$

$$\mathbf{d}^T = \begin{bmatrix} 0 & 0 & 21.0 \\ 0 & 0 & 1.5 \\ 0 & 0 & -32.5 \\ 0 & -23 & 0 \\ -27 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times10^{-12}\ \mathrm{c/N},\quad \frac{\varepsilon}{\varepsilon_0} = \begin{bmatrix} 6.1 & 0 & 0 \\ 0 & 7.5 & 0 \\ 0 & 0 & 6.7 \end{bmatrix}$$

$$\mathbf{e}^T = \mathbf{c}^E\mathbf{d}^T,\quad \alpha = 120\times10^{-6}\ \mathrm{K}^{-1},\quad \rho = 1800\ \mathrm{kg/m^3}$$

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